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LETTER TO THE EDITOR

Grammian N-soliton solutions of a coupled KdV system

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Abstract
From the Lax pair and the binary Darboux transformation of a coupled Korteweg–de Vries system, we show that its nonlinear superposition formula is identical to that obtained for the Kaup–Kupershmidt equation. Therefore, the N-soliton solution can be associated with a determinant of the Gram type.

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1. Introduction

In [1], Karasu and Sakovich performed the Painlevé analysis of the system of two coupled nonlinear evolution equations of Korteweg–de Vries (KdV) type

$$u_t - \frac{a}{4}u_{xxx} - buu_x - \frac{9a^2h}{4b}v_x = 0 \qquad v_t + \frac{a}{2}v_{xxx} + buv_x = 0 \qquad a, b, h \text{ constants } \neq 0 \tag{1}$$

which, in setting $u = \frac{3a}{b}w_x$ and eliminating v, is equivalent to the sixth-order equation

$$\frac{4}{3a}w_{tt} + \frac{1}{3}w_{xxxt} - \frac{a}{6}w_{xxxxxx} - 3aw_xw_{xxxx} - 6aw_{xx}w_{xxx} - 12aw_x^2w_{xx} = 0. \tag{2}$$

They showed that the equation passes the Painlevé test, and using the truncation method of Weiss *et al* [2] derived a Bäcklund transformation (BT). Then they built special solutions generated by this transformation from the vacuum $w = 0$.

The system (1) was found by Satsuma and Hirota [5] as a special case of the four-reduction of the KP hierarchy; more precisely, it is the system numbered in their paper as (4.18a), (4.18b). (It was also proposed by Drinfel'd and Sokolov [3] and Bogoyavlenskii [4].) Moreover, Satsuma and Hirota gave for (1) the expression of the one-soliton

$$u = \frac{3a}{b}w_x \qquad w = \partial_x \log f \qquad v = \frac{1}{h}\left(\frac{4}{3a}w_t - \frac{1}{3}w_{xxx} - 2w_x^2\right) \tag{3}$$

$$f = 1 + 2e^{\theta} + \tfrac{1}{2}e^{2\theta} \qquad \theta = \kappa x - \frac{a}{2}\kappa^3 t + \delta \qquad \kappa \text{ and } \delta \text{ arbitrary constants} \tag{4}$$

as well as the fourth-order Lax pair

$$\frac{\partial\psi}{\partial t} = \left(a\partial_x^3 + bu\partial_x + \frac{b}{2}u_x\right)\psi \tag{5}$$

$$\lambda^4\psi\left(\partial_x^4 + \frac{4b}{3a}u\partial_x^2 + \frac{4b}{3a}u_x\partial_x + \frac{2b}{3a}u_{xx} + \frac{4b^2}{9a^2}u^2 + hv\right)\psi = \lambda^4\psi. \tag{6}$$

In their paper, Karasu and Sakovich noticed that their BT is probably not the simplest one for the system (1) for the reason that they apparently could not recover the expression (4) of the one-soliton solution.

In this Letter we show that by considering the Darboux transformation (DT)

$$w = \partial_x \log f + W \qquad \text{with} \quad f = \int^x \psi^2\,\mathrm{d}x \tag{7}$$

where w and W are two solutions of (2) and ψ a solution of the Lax pair (5), (6), eliminating ψ between the DT and the Lax pair, we obtain the same BT as Karasu and Sakovich. Furthermore, remarking that one of the two equations determining the BT can be associated with the Gambier 25 equation [6], as we did previously [7, 8] for the Kaup–Kupershmidt partial differential equation, we easily derive the nonlinear superposition formula (NLSF) for equation (2) and show that the N-soliton solution is related to the logarithmic derivative of a determinant of the Gram type, as introduced by Nakamura [9] for the KP equation.

We also show that the rational solutions given by Karasu and Sakovich are trivially retrieved by setting in (5), (6) $u = v = 0$ and $\lambda = 0$, taking account of the DT (7), and that for $\lambda \neq 0$ their last solution corresponds to the one-soliton solution (4) if one makes an appropriate choice of the constants of integration.

Finally, we indicate the construction of the N-soliton solution and give the explicit expression of the two-soliton.

2. Bäcklund transformation and nonlinear superposition formula

The elimination of ψ between the DT (7) and the Lax pair (5), (6), taking into account that (5) possesses the first integral

$$\int^x (\psi^2)_t\,\mathrm{d}x = 2a\psi\psi_{xx} - a\psi_x^2 + bu\psi^2 \tag{8}$$

yields the BT

$$p_t = a\left(p_{xx} - \frac{3}{4}\frac{p_x^2}{p} + \frac{3}{2}pp_x + 3pW_x + \frac{1}{4}p^3\right)_x \qquad p = w - W \tag{9}$$

$$\begin{aligned}\frac{1}{2}p_{xxxx} &+ \frac{5}{3}W_{xxx}p - \frac{p_x p_{xxx}}{p} + \frac{3}{2}pp_{xxx} + 2p_xW_{xx} + 2p^2W_{xx} - \frac{3}{4}\frac{p_{xx}^2}{p} + 2W_xp_{xx}\\ &+\frac{9}{4}\frac{p_x^2p_{xx}}{p^2} + p_xp_{xx} + \frac{7}{4}p^2p_{xx} + 2pW_x^2 - \frac{p_x^2W_x}{p} + 4pp_xW_x + p^3W_x\\ &-\frac{15}{16}\frac{p_x^4}{p^3} + \frac{15}{8}pp_x^2 + p^3p_x + \frac{4}{3a}pW_t + \frac{1}{16}p^5 = \lambda^4 p\end{aligned} \tag{10}$$

and if one uses (9) to eliminate the highest derivative in (10), we obtain exactly the same BT as Karasu and Sakovich.

Now, let us remark that the integral of the right-hand side of equation (9) can be identified with the nonlinear ordinary differential equation G25 of the Gambier classification [6], possessing the Painlevé property and already associated with the BT of the Kaup–Kupershmidt equation in [7]. Therefore, considering four copies of the equation (9) for $p = w_{12} - w_2$, $w_{12} - w_1$, $w_2 - w_0$, $w_1 - w_0$, i.e.

$$(w_{12} - w_2)_t = a\left((w_{12} - w_2)_{xx} - \frac{3}{4}\frac{(w_{12} - w_2)_x^2}{(w_{12} - w_2)} + \frac{3}{2}(w_{12} - w_2)(w_{12} - w_2)_x \right.$$
$$\left. +3(w_{12} - w_2)w_{2,x} + \tfrac{1}{4}(w_{12} - w_2)^3\right)_x$$
$$(w_{12} - w_1)_t = a\left((w_{12} - w_1)_{xx} - \frac{3}{4}\frac{(w_{12} - w_1)_x^2}{(w_{12} - w_1)} + \frac{3}{2}(w_{12} - w_1)(w_{12} - w_1)_x \right.$$
$$\left. +3(w_{12} - w_1)w_{1,x} + \tfrac{1}{4}(w_{12} - w_1)^3\right)_x$$
$$(w_2 - w_0)_t = a\left((w_2 - w_0)_{xx} - \frac{3}{4}\frac{(w_2 - w_0)_x^2}{(w_2 - w_0)} + \frac{3}{2}(w_2 - w_0)(w_2 - w_0)_x \right.$$
$$\left. +3(w_2 - w_0)w_{0,x} + \tfrac{1}{4}(w_2 - w_0)^3\right)_x$$
$$(w_1 - w_0)_t = a\left((w_1 - w_0)_{xx} - \frac{3}{4}\frac{(w_1 - w_0)_x^2}{(w_1 - w_0)} + \frac{3}{2}(w_1 - w_0)(w_1 - w_0)_x \right.$$
$$\left. +3(w_1 - w_0)w_{0,x} + \tfrac{1}{4}(w_1 - w_0)^3\right)_x$$

and making the combination which eliminates the linear terms, we can integrate once with respect to x and obtain the first-order, second-degree ordinary differential equation:

$$(w_{12,x} + w_{12}^2 - Aw_{12} + 2B)^2 - \frac{C^2(w_{12} - w_1)(w_{12} - w_2)}{(w_1 - w_0)(w_2 - w_0)} = \frac{4}{3}K\frac{(w_{12} - w_1)(w_{12} - w_2)}{w_2 - w_1} \tag{11}$$

with coefficients

$$A = (w_1 + w_2) + \frac{w_{2,x} - w_{1,x}}{w_2 - w_1} \qquad B = \frac{1}{2}w_1 w_2 + \frac{w_{2,x}w_1 - w_{1,x}w_2}{2(w_2 - w_1)} \tag{12}$$
$$C = w_{0,x} + w_0^2 - Aw_0 + 2B \tag{13}$$

where K is a constant of integration.

Setting $K = 0$, $w_i = \partial_x \log f_i$, $i = 0, 1, 2, 12$ and defining

$$F_{12} = \frac{f_{12}}{f_0} \qquad F_2 = \frac{f_2}{f_0} \qquad F_1 = \frac{f_1}{f_0} \tag{14}$$

we obtain the third-order linear equation for F_{12}

$$F_{12,xxx} - \left(D\frac{(F_1F_2)_x}{F_{1,x}F_{2,x}} + \frac{F_{1,x}F_{2,xxx} - F_{2,x}F_{1,xxx}}{F_{1,x}F_{2,xx} - F_{2,x}F_{1,xx}}\right)F_{12,xx}$$
$$+\left(D\frac{F_1F_{2,xx} + F_2F_{1,xx}}{F_{1,x}F_{2,x}} + \frac{F_{1,xx}F_{2,xxx} - F_{2,xx}F_{1,xxx}}{F_{1,x}F_{2,xx} - F_{2,x}F_{1,xx}}\right)F_{12,x} = 0 \tag{15}$$

$$D = \frac{W(F_{2,x}, F_{1,x})}{2\,W(F_2, F_1)} \qquad \text{with} \quad W(a, b) = a_x b - a b_x \tag{16}$$

which possesses the general solution

$$F_{12} = K_1(t) + K_2(t)R_{12} + K_3(t)\left(F_1F_2 - R_{12}^2\right) \tag{17}$$
$$R_{12} = \int^x \sqrt{F_{1,x}F_{2,x}}\,\mathrm{d}x. \tag{18}$$

Setting $K_1 = K_2 = 0$ and taking account of the definitions (14) and (7), the NLSF for (2) is

$$f_{12} = f_0 \begin{vmatrix} \int^x \psi_1^2 & \int^x \psi_1\psi_2 \\ \int^x \psi_1\psi_2 & \int^x \psi_2^2 \end{vmatrix} \tag{19}$$

which is exactly the same expression as the NLSF for the Kaup–Kupershmidt equation [8]. Therefore, taking into account for the construction of the N-soliton solution that the seed solution is $f_0 = 1$, one may iterate the formula (19) [10] and obtain the N-soliton solution:

$$f^{(N)} = \det\left[\int^x \psi_i\psi_j\,\mathrm{d}x\right]_{1\leqslant i,j\leqslant N} \tag{20}$$

where ψ_i is the vacuum wavefunction, solution of the system (5), (6) for $v = u = 0$, $\lambda = \lambda_i$.

3. Construction of the N-soliton

For $\lambda = 0$, the vacuum wavefunction ψ_0 is a third-degree polynomial in x

$$\psi_0 = c_1x^3 + c_2x^2 + c_3x + 6ac_1t + c_4 \tag{21}$$

and following the values given to the arbitrary constants c_1, c_2, c_3, c_4 one easily generates with the DT (7), setting $W = 0$, the solutions numbered (9)–(12) in the paper of Karasu and Sakovich corresponding respectively to the choice of parameters c_i: $c_1 = c_2 = c_3 = 0$, $c_1 = c_2 = c_4 = 0$, $c_1 = c_3 = 0$, $c_2 = c_4 = 0$.

For $\lambda \neq 0$, the vacuum wavefunction ψ_k is a superposition of four exponentials:

$$\psi_k = A_k\mathrm{e}^{\lambda_kx+a\lambda_k^3t} + B_k\mathrm{e}^{-\lambda_kx-a\lambda_k^3t} + C_k\mathrm{e}^{\mathrm{i}\lambda_kx-\mathrm{i}a\lambda_k^3t} + D_k\mathrm{e}^{-\mathrm{i}\lambda_kx+\mathrm{i}a\lambda_k^3t} \qquad i^2 = -1 \tag{22}$$

and one has that

$$\begin{aligned} f_k = \int^x \psi_k^2\,\mathrm{d}x &= \frac{A_k^2}{2\lambda_k}\mathrm{e}^{2(\lambda_kx+a\lambda_k^3t)} - \frac{B_k^2}{2\lambda_k}\mathrm{e}^{-2(\lambda_kx+a\lambda_k^3t)} - \frac{\mathrm{i}C_k^2}{2\lambda_k}\mathrm{e}^{2\mathrm{i}(\lambda_kx-a\lambda_k^3t)} + \frac{\mathrm{i}D_k^2}{2\lambda_k}\mathrm{e}^{-2\mathrm{i}(\lambda_kx-a\lambda_k^3t)} \\ &+\frac{(1-\mathrm{i})A_kC_k}{\lambda_k}\mathrm{e}^{(1+\mathrm{i})\lambda_kx+(1-\mathrm{i})a\lambda_k^3t} + \frac{(1+\mathrm{i})A_kD_k}{\lambda_k}\mathrm{e}^{(1-\mathrm{i})\lambda_kx+(1+\mathrm{i})a\lambda_k^3t} \\ &-\frac{(1+\mathrm{i})B_kC_k}{\lambda_k}\mathrm{e}^{-(1-\mathrm{i})\lambda_kx-(1+\mathrm{i})a\lambda_k^3t} - \frac{(1-\mathrm{i})B_kD_k}{\lambda_k}\mathrm{e}^{-(1+\mathrm{i})\lambda_kx-(1-\mathrm{i})a\lambda_k^3t} \\ &+2(A_kB_k + C_kD_k)x + 6a\lambda_k^2(A_kB_k - C_kD_k)t \end{aligned} \tag{23}$$

which yields the solution (13) in the paper of Karasu and Sakovich. To build the N-soliton solution, one considers the particular case

$$A_k = C_k = 0. \tag{24}$$

We first derive the expression of the one-soliton solution. Setting $\lambda_k \equiv \lambda$, $B_k \equiv B$, $D_k \equiv D$ in (23), we have

$$f = -\frac{B^2}{2\lambda}\mathrm{e}^{-2(\lambda x+a\lambda^3t)}\left(1 - \frac{\mathrm{i}D^2}{B_j^2}\mathrm{e}^{2(1-\mathrm{i})\lambda x+2a(1+\mathrm{i})\lambda^3t} + \frac{4D}{(1+\mathrm{i})B}\mathrm{e}^{(1-\mathrm{i})\lambda x+a(1+\mathrm{i})\lambda^3t}\right). \tag{25}$$

Therefore, up to an exponential linear in x and t one has that

$$f = 1 + 2\mathrm{e}^{\theta} + \tfrac{1}{2}\mathrm{e}^{2\theta} \qquad \theta = \kappa x - \frac{a}{2}\kappa^3t + \delta \quad \delta = \log\frac{2D}{(1+\mathrm{i})B} \quad \kappa = (1-\mathrm{i})\lambda. \tag{26}$$

Using the expression (22) and the relation (19) for $k = 1, 2$ in the formula (24), one obtains for the two-soliton solution

$$
\begin{aligned}
f_{12} = 1 + 2\mathrm{e}^{\theta_1} + 2\mathrm{e}^{\theta_2} + \frac{1}{2}\mathrm{e}^{2\theta_1} + \frac{1}{2}\mathrm{e}^{2\theta_2} + \frac{4(\kappa_1^4 + \kappa_2^4)}{(\kappa_1 + \kappa_2)^2(\kappa_1^2 + \kappa_2^2)}\mathrm{e}^{\theta_1+\theta_2} \\
+ \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2}(\mathrm{e}^{2\theta_1+\theta_2} + \mathrm{e}^{\theta_1+2\theta_2}) + \frac{(\kappa_1 - \kappa_2)^4}{4(\kappa_1 + \kappa_2)^4}\mathrm{e}^{2(\theta_1+\theta_2)} \\
\theta_j = \kappa_j - \frac{a}{2}\kappa_j^3 t + \delta_j \qquad \delta_j = \frac{C_j(1+\mathrm{i})(-\mathrm{i}\kappa_j + \kappa_l)(\kappa_j + \kappa_l)}{B_j(\kappa_j - \kappa_l)(\kappa_j - \mathrm{i}k_l)} \\
1 \leqslant j, l \leqslant 2 \quad l \neq j \quad \kappa_j = (1 - \mathrm{i})\lambda_j.
\end{aligned}
\tag{27}
$$

To build the N-soliton solution, we set (24) in (20) for $1 \leqslant j, l \leqslant N$.

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